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A simple construction of fermion measure term in U(1) chiral lattice gauge theories with exact gauge invariance

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Abstract: In the gauge invariant formulation of U(1) chiral lattice gauge theories based on the Ginsparg-Wilson relation, the gauge field dependence of the fermion measure is determined through the so-called measure term. We derive a closed formula of the measure term on the finite volume lattice. The Wilson line degrees of freedom (torons) of the link field are treated separately to take care of the global integrability. The local counter term is explicitly constructed with the local current associated with the cohomologically trivial part of the gauge anomaly in finite volume. The resulted formula is very close to the known expression of the measure term in infinite volume with a single parameter integration, and would be useful in practical implementations.

Keywords: Lattice Gauge Field Theories, Anomalies in Field and String Theories, Gauge Symmetry, Standard Model.

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1. Introduction

Chiral gauge theories have several interesting possibilities in their own dynamics: fermion number non-conservation due to chiral anomaly [1, 2], various realizations of the gauge symmetry and global flavor symmetry [3, 4], the existence of massless composite fermions suggested by 't Hooft's anomaly matching condition [5] and so on. Unfortunately, very little is known so far about the actual behavior of chiral gauge theories beyond perturbation theory. It is desirable to develop a formulation to study the non-perturbative dynamics of chiral gauge theories.

Lattice gauge theory can now provide a framework for non-perturbative formulation of chiral gauge theories. The clue to this development is the construction of gauge-covariant

and local lattice Dirac operators satisfying the Ginsparg-Wilson relation [6 – 11].[1] By this relation, it is possible to realize an exact chiral symmetry on the lattice [34]. It is also possible to introduce Weyl fermions on the lattice and this opens the possibility to formulate anomaly-free chiral lattice gauge theories [35 – 48]. Although it is believed that the chiral gauge theory is a difficult case for numerical simulations because the effective action induced by Weyl fermions has a non-zero imaginary part, still it would be interesting and even useful to develop a formulation of chiral lattice gauge theories by which one can work out fermionic observables numerically as the functions of link field with exact gauge invariance.

In the case of U(1) chiral gauge theories, Lüscher [36] proved rigorously that it is possible to construct the fermion path-integral measure which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries.[2] In this formulation, however, although the proof of the existence of the fermion measure is constructive, the resulted formula of the fermion measure turns out to be rather complicated for the case of the finite-volume lattice. In particular, to take into account the requirements of locality and smoothness, it is based on the procedure to separate the part definable in infinite volume and the part of the finite volume corrections. Therefore it does not provide a formulation which is immediately usable for numerical applications. The purpose of this paper is to present a simple and closed expression of the fermion measure (term) for the U(1) chiral lattice gauge theories defined within the finite-volume lattice.

This paper is organized as follows. In section 2, we review the construction of U(1) chiral lattice gauge theories based on the Ginsparg-Wilson relation and the reconstruction theorem of the fermion measure formulated by Lüscher [36]. In section 3, we describe our construction of the fermion measure term on the finite-volume lattice, which fulfills all the required properties for the reconstruction theorem. In our formulation, the Wilson line degrees of freedom of the link field (torons) are treated separately: we first construct the measure term for these degrees of freedom to take care of the global integrability. The part of local counter term is then explicitly constructed with the local current associated with the cohomologically trivial part of the gauge anomaly in finite volume. Combining these results, we finally obtain a closed formula of the measure term on the finite volume lattice, which is similar to the known expression of the measure term in the infinite volume with

[1] An explicit solution of the Ginsparg-Wilson relation was derived from the overlap formalism proposed by Narayanan and Neuberger [12 – 22] and is referred as the overlap Dirac operator. The overlap formalism gives a well-defined partition function of Weyl fermions on the lattice, which nicely reproduces the fermion zero mode and the fermion-number violating observables ('t Hooft vertices) [23 – 25]. Through the recent re-discovery of the Ginsparg-Wilson relation, the meaning of the overlap formula, especially the locality properties, become clear from the point of view of the path-integral. For Dirac fermions, the overlap formalism provides a gauge-covariant and local lattice Dirac operator satisfying the Ginsparg-Wilson relation [6, 7, 21, 9, 11]. The overlap formula was derived from the five-dimensional approach of domain wall fermion proposed by Kaplan [26]. In the vector-like formalism of domain wall fermion [27 – 30], the local low energy effective action of the chiral mode precisely reproduces the overlap Dirac operator [31 – 33].

[2] The gauge-invariant construction by Lüscher [36] based on the Ginsparg-Wilson relation provides a procedure to determine the phase of the overlap formula in a gauge-invariant manner for anomaly-free U(1) chiral gauge theories.

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one parameter integration. Section 4 is devoted to summary and discussions.

2. U(1) chiral gauge theories on the lattice with exact gauge invariance

In this section, we review the construction of U(1) chiral lattice gauge theories with exact gauge invariance given by Lüscher [36]. We consider U(1) gauge theories where the gauge field couples to N left-handed Weyl fermions with charges e_α satisfying the anomaly cancellation condition,

$$\sum_{\alpha=1}^{N} \mathrm{e}_\alpha^3 = 0. \tag{2.1}$$

We assume the four-dimensional lattice of the finite size L and choose lattice units,

$$\Gamma = \left\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid 0 \le x_\mu < L \, (\mu = 1, 2, 3, 4) \right\}, \tag{2.2}$$

and adopt the periodic boundary condition for both boson fields and fermion fields.

2.1 Gauge fields

We adopt the compact formulation of U(1) gauge theory on the lattice. U(1) gauge fields on Γ then are represented by link fields, $U(x, \mu) \in \mathrm{U}(1)$. We require the so-called admissibility condition on the gauge fields:

$$|F_{\mu\nu}(x)| < \epsilon \qquad \text{for all} \quad x, \mu, \nu, \tag{2.3}$$

where the field tensor $F_{\mu\nu}(x)$ is defined from the plaquette variables,

$$F_{\mu\nu}(x) = \frac{1}{i} \ln P_{\mu\nu}(x), \quad -\pi < F_{\mu\nu}(x) \le \pi, \tag{2.4}$$

$$P_{\mu\nu}(x) = U(x, \mu) U(x + \hat{\mu}, \nu) U(x + \hat{\nu}, \mu)^{-1} U(x, \nu)^{-1}, \tag{2.5}$$

and ϵ is a fix number in the range $0 < \epsilon < \pi/3$. This condition ensures that the overlap Dirac operator [7, 9] is a smooth and local function of the gauge field if $|\mathrm{e}_\alpha|\epsilon < 1/30$ for all α [11]. The admissibility condition may be imposed dynamically by choosing the following action,

$$S_G = \frac{1}{4g_0^2} \sum_{x \in \Gamma} \sum_{\mu,\nu} L_{\mu\nu}(x), \tag{2.6}$$

where

$$L_{\mu\nu}(x) = \begin{cases} [F_{\mu\nu}(x)]^2 \left\{ 1 - [F_{\mu\nu}(x)]^2 / \epsilon^2 \right\}^{-1} & \text{if } |F_{\mu\nu}(x)| < \epsilon, \\ \infty & \text{otherwise.} \end{cases} \tag{2.7}$$

The admissible U(1) gauge fields can be classified by the magnetic fluxes,

$$m_{\mu\nu} = \frac{1}{2\pi} \sum_{s,t=0}^{L-1} F_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu}), \tag{2.8}$$

which are integers independent of x. We denote the space of the admissible gauge fields with a given magnetic flux $m_{\mu\nu}$ by $\mathfrak{U}[m]$. As a reference point in the given topological

sector $\mathfrak{U}[m]$, one may introduce the gauge field which has the constant field tensor equal to $2\pi m_{\mu\nu}/L^2 (< \epsilon)$ by

$$V_{[m]}(x,\mu) = \mathrm{e}^{-\frac{2\pi i}{L^2}\left[L\delta_{\tilde{x}_\mu,L-1}\sum_{\nu>\mu} m_{\mu\nu}\tilde{x}_\nu + \sum_{\nu<\mu} m_{\mu\nu}\tilde{x}_\nu\right]} \quad (\tilde{x}_\mu = x_\mu \bmod L). \tag{2.9}$$

Then any admissible U(1) gauge field in $\mathfrak{U}[m]$ may be expressed as

$$U(x,\mu) = \tilde{U}(x,\mu)\, V_{[m]}(x,\mu), \tag{2.10}$$

where $\tilde{U}(x,\mu)$ stands for the dynamical degrees of freedom. Accordingly, any local variation of the link field $U(x,\mu) \in \mathfrak{U}[m]$ should refer to $\tilde{U}(x,\mu)$:

$$\delta U(x,\mu) = \left\{\delta\tilde{U}(x,\mu)\right\} V_{[m]}(x,\mu). \tag{2.11}$$

U(1) gauge fields on Γ with the periodic boundary condition may be represented through periodic link fields on the infinite lattice:

$$U(x,\mu) \in \mathrm{U}(1), \qquad x \in \mathbb{Z}^4, \tag{2.12}$$

$$U(x + L\hat{\nu},\mu) = U(x,\mu) \qquad \text{for all } \mu,\nu. \tag{2.13}$$

2.2 Weyl fields

Weyl fermions are introduced based on the Ginsparg-Wilson relation. We first consider Dirac fields $\psi(x)$ which carry a Dirac index and a flavor index $\alpha = 1,\ldots,N$. Each component $\psi_\alpha(x)$ couples to the link field, $U(x,\mu)^{\mathrm{e}_\alpha}$. We assume that the lattice Dirac operator acting on $\psi(x)$ satisfies the Ginsparg-Wilson relation,[3]

$$\gamma_5 D_L + D_L\hat{\gamma}_5 = 0, \qquad \hat{\gamma}_5 \equiv \gamma_5(1 - 2D_L), \tag{2.14}$$

and we define the projection operators as

$$\hat{P}_\pm = \left(\frac{1 \pm \hat{\gamma}_5}{2}\right), \quad P_\pm = \left(\frac{1 \pm \gamma_5}{2}\right). \tag{2.15}$$

The left-handed Weyl fermions, for example, can be defined by imposing the constraints,

$$\psi_-(x) = \hat{P}_-\psi(x), \quad \bar{\psi}_-(x) = \bar{\psi}(x)P_+. \tag{2.16}$$

The action of the left-handed Weyl fermions is then given by

$$S_W = \sum_{x\in\Gamma} \bar{\psi}_-(x) D_L \psi_-(x). \tag{2.17}$$

The kernel of the lattice Dirac operator in finite volume, D_L, may be represented through the kernel of the lattice Dirac operator in infinite volume, D, as follows:

$$D_L(x,y) = D(x,y) + \sum_{n\in\mathbb{Z}^4, n\neq 0} D(x, y+nL), \tag{2.18}$$

[3]In this paper, we adopt the normalization of the lattice Dirac operator so that the factor 2 appears in the right-hand-side of the Ginsparg-Wilson relation: $\gamma_5 D_L + D_L\gamma_5 = 2D_L\gamma_5 D_L$.

where $D(x,y)$ is defined with a periodic link field in infinite volume. We assume that $D(x,y)$ posseses the locality property given by

$$\|D(x,y)\| \leq C(1+\|x-y\|^p)\,\mathrm{e}^{-\|x-y\|/\varrho} \tag{2.19}$$

for some constants $\varrho > 0$, $C > 0$, $p \geq 0$, where ϱ is the localization range of the lattice Dirac operator.

2.3 Path-integral measure of Weyl fermions

The path-integral measure of the Weyl fermions may be defined by the Grassmann integrations,

$$\mathcal{D}[\psi_-]\mathcal{D}[\bar{\psi}_-] = \prod_j dc_j \prod_k d\bar{c}_k, \tag{2.20}$$

where $\{c_j\}$ and $\{\bar{c}_k\}$ are the grassman coefficients in the expansion of the Weyl fields,

$$\psi_-(x) = \sum_j v_j(x)c_j, \quad \bar{\psi}_-(x) = \sum_k \bar{c}_k \bar{v}_k(x) \tag{2.21}$$

in terms of the chiral (orthonormal) basis defined by

$$\hat{P}_- v_j(x) = v_j(x), \quad \bar{v}_k(x)P_+ = \bar{v}_k(x). \tag{2.22}$$

Since the projection operator $\hat{P}_-$ depends on the gauge field through D, the fermion measure also depends on the gauge field. In this gauge-field dependence of the fermion measure, there is an ambiguity by a pure phase factor, because any unitary transformation of the basis,

$$\tilde{v}_j(x) = \sum_l v_l(x)\left(\mathcal{Q}^{-1}\right)_{lj}, \qquad \tilde{c}_j = \sum_l \mathcal{Q}_{jl}c_l, \tag{2.23}$$

induces a change of the measure by the pure phase factor $\det \mathcal{Q}$. This ambiguity should be fixed so that it fulfills the fundamental requirements such as locality, gauge-invariance, integrability and lattice symmetries.

2.4 Reconstruction theorem of the Weyl fermion measure

The properties of the fermion measure can be characterized by the so-called measure term which is given in terms of the chiral basis and its variation with respect to the gauge field, $\delta_\eta U(x,\mu) = i\eta_\mu(x)U(x,\mu)$, as

$$\mathfrak{L}_\eta = i\sum_j (v_j, \delta_\eta v_j). \tag{2.24}$$

The reconstruction theorem given in [36] asserts that if there exists a local current $j_\mu(x)$ which satisfies the following four properties, it is possible to reconstruct the fermion measure (the basis $\{v_j(x)\}$) which depends smoothly on the gauge field and fulfills the fundamental requirements such as locality,[4] gauge-invariance, integrability and lattice symmetries:[5]

[4]We adopt the generalized notion of locality on the lattice given in [11, 35, 36] for Dirac operators and composite fields. See also [57] for the case of the finite volume lattice.

[5]The lattice symmetries mean translations, rotations, reflections and charge conjugation.

Theorem 1. *Suppose $j_\mu(x)$ is a given current with the following properties:*[6]

1. *$j_\mu(x)$ is defined for all admissible gauge fields and depends smoothly on the link variables.*

2. *$j_\mu(x)$ is gauge-invariant and transforms as an axial vector current under the lattice symmetries.*

3. *The linear functional $\mathfrak{L}_\eta = \sum_{x\in\Gamma} \eta_\mu(x) j_\mu(x)$ is a solution of the integrability condition*

$$\delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta = i\mathrm{Tr}_L \left\{ \hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} \tag{2.25}$$

   for all periodic variations $\eta_\mu(x)$ and $\zeta_\mu(x)$.

4. *The anomalous conservation law holds:*

$$\partial^*_\mu j_\mu(x) = \mathrm{tr}\{Q\gamma_5(1 - D_L)(x,x)\}, \qquad Q = diag(\mathrm{e}_1, \ldots, \mathrm{e}_N). \tag{2.26}$$

Then there exists a smooth fermion integration measure in the vacuum sector such that the associated current coincides with $j_\mu(x)$. The same is true in all other sectors if the number of fermion flavors with $|\mathrm{e}_\alpha| = \mathrm{e}$ is even for all odd e. *In each case the measure is uniquely determined up to a constant phase factor.*

A comment is in order about the topological aspects of the reconstrtuction theorem. As discussed in [36], it is possible to associate a U(1) bundle with the fermion measure. In this point of view, the measure term, $\mathfrak{L}_\eta$ defined by eq. (2.24), can be regarded as the connection of the U(1) bundle, and the quantity which appears in the r.h.s. of the integrability condition eq. (2.25),

$$\mathfrak{C}_{\eta\zeta} \equiv i\mathrm{Tr}_L \left\{ \hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\}, \tag{2.27}$$

is nothing but the curvature of the connection,

$$\mathfrak{C}_{\eta\zeta} = \delta_\eta \mathfrak{L}_\zeta - \delta_\zeta \mathfrak{L}_\eta. \tag{2.28}$$

It is known that the integration of the curvature of a U(1) bundle over any two-dimensional closed surface in the base manifold takes value of the multiples of 2π. If one parametrize a two-dimensional closed surface in the space of the admissible U(1) gauge fields by $s, t \in [0, 2\pi]$, then one has

$$\int_0^{2\pi} ds \int_0^{2\pi} dt \; i\mathrm{Tr} \left\{ \hat{P}_-[\partial_s \hat{P}_-, \partial_t \hat{P}_-] \right\} = 2\pi \times \text{integer}. \tag{2.29}$$

If (and only if) the U(1) bundle is trivial, these integrals of the curvature vanishes identically. The integrability condition eq. (2.25) asserts that it is indeed the case and the fermion measure is then smooth.

[6] Throughout this paper, $\mathrm{Tr}\{\cdots\}$ stands for the trace over the lattice index x, the flavor index $\alpha(= 1, \ldots, N)$ and the spinor index, while tr stands for the trace over the flavor and spinor indices only. $\mathrm{Tr}_L\{\cdots\}$ stands for the trace over the finite lattice, $x \in \Gamma$.

2.5 Constructive proof of the existence of the measure term

In [36], it is proved constructively that there exists a *local* current $j_\mu(x)$ which satisfies the properties required in the reconstruction theorem. In fact, the construction of the current is not straightforward by two reasons. The first reason is that the measure term must be smooth w.r.t. the gauge field, but the topology of the space of the admissible gauge fields in finite volume is not trivial. The second reason is that the locality property of the current must be maintained even in finite volume. To take these points into account, the construction in [36] is made in two steps by sperating the part definable in infinite volume from the part of the finite volume corrections.

The procedure to sperate the part definable in infinite volume from the part of the finite volume corrections is as follows. As eq. (2.18), one may represent the kernel of the chiral projector in finite volume $\hat{P}_-(x,y)$ through the kernel of chiral projector in infinite volume, $P(x,y) = \frac{1}{2}(1-\gamma_5)\delta_{xy} + \frac{1}{2}\gamma_5 D(x,y)$, as

$$\hat{P}_-(x,y) = \sum_{n\in\mathbb{Z}^4} P(x, y+nL). \tag{2.30}$$

One may also introduce the projector Q_Γ acting on the fields in infinite volume as

$$Q_\Gamma\,\psi(x) = \begin{cases} \psi(x) & \text{if } x \in \Gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{2.31}$$

Using these, the right-hand-sides of the integrability condition eq. (2.25) and the anomalous conservation law eq. (2.26) may be rewritten into

$$i\mathrm{Tr}_L\left\{\hat{P}_-[\delta_\eta\hat{P}_-, \delta_\zeta\hat{P}_-]\right\} = i\mathrm{Tr}\left\{Q_\Gamma\hat{P}_-[\delta_\eta\hat{P}_-, \delta_\zeta\hat{P}_-]\right\} + \mathfrak{R}_{\eta\zeta}, \tag{2.32}$$

and

$$\mathrm{tr}\{Q\gamma_5(1-D_L)(x,x)\} = \mathrm{tr}\{Q\gamma_5(1-D)(x,x)\} + r(x), \tag{2.33}$$

respectively, where $\mathfrak{R}_{\eta\zeta}$ and $r(x)$ are finite-volume corrections,

$$\begin{aligned}\mathfrak{R}_{\eta\zeta} = i\sum_{x\in\Gamma}\sum_{y,z\in\mathbb{Z}^4}\sum_{n\in\mathbb{Z}^4, n\neq 0} &\mathrm{tr}\,\{P(x,y) \\ &\times [\delta_\eta P(y,z)\delta_\zeta P(z,x+Ln) - \delta_\zeta P(y,z)\delta_\eta P(z,x+Ln)]\}\,,\end{aligned} \tag{2.34}$$

$$r(x) = \sum_{n\in\mathbb{Z}^4, n\neq 0} \mathrm{tr}\{Q\gamma_5(1-D)(x,x+Ln)\}, \tag{2.35}$$

satisfying

$$|\mathfrak{R}_{\eta\zeta}| \le \kappa_1 L^{\nu_1} \mathrm{e}^{-L/\varrho}\|\eta\|_\infty\|\zeta\|_\infty, \tag{2.36}$$

$$|r(x)| \le C_1\,\mathrm{e}^{-L/\varrho} \tag{2.37}$$

for some constants $\kappa_1 > 0$, $\nu_1 \ge 0$ and $C_1 > 0$. These bounds follow from the locality property of the lattice Dirac operator D in infinite volume eq. (2.19).

Then, as the first step, one constructs a local current $j^\star_\mu(x)$ in infinite volume so that 1) it depends smoothly on the link variables, 2) it is gauge-invariant and transforms as an axial vector current under the lattice symmetries, 3) the linear functional defined with a periodic link variables,

$$\mathfrak{K}_\eta = \sum_{x\in\Gamma} \eta_\mu(x) j^\star_\mu(x), \tag{2.38}$$

is a solution of the integrability condition

$$\delta_\eta \mathfrak{K}_\zeta - \delta_\zeta \mathfrak{K}_\eta = i\mathrm{Tr}\left\{ Q_\Gamma \hat{P}_-[\delta_\eta \hat{P}_-, \delta_\zeta \hat{P}_-] \right\} \tag{2.39}$$

for all periodic variations $\eta_\mu(x)$ and $\zeta_\mu(x)$, and 4) it satisfies the anomalous conservation law in infinite volume,

$$\partial_\mu j^\star_\mu(x) = \mathrm{tr}\{Q\gamma_5(1-D)(x,x)\}. \tag{2.40}$$

As the second step, one constructs the finite-volume correction to $\mathfrak{K}_\eta$,

$$\mathfrak{S}_\eta = \sum_{x\in\Gamma} \eta_\mu(x) \Delta j_\mu(x) \tag{2.41}$$

with the property

$$|\Delta j_\mu(x)| \le \kappa_2 L^{\nu_2} \mathrm{e}^{-L/\varrho} \tag{2.42}$$

for some constants $\kappa_2 > 0$, $\nu_2 \ge 0$, so that it satisfies the conditions 1) and 2) above and

$$\delta_\eta \mathfrak{S}_\zeta - \delta_\zeta \mathfrak{S}_\eta = \mathfrak{R}_{\eta\zeta}, \qquad \partial_\mu \Delta j_\mu(x) = r(x). \tag{2.43}$$

The linear functional $\mathfrak{L}_\eta \equiv \mathfrak{K}_\eta + \mathfrak{S}_\eta$ then fulfills all the required properties for the measure term *on the finite-volume lattice.*[7]

2.5.1 First step in infinite volume: locality

In the first step, the explicit expression of the local current $j^\star_\mu(x)$ is obtained [36]. This is based on the two facts which hold true in infinite volume.

The first fact is about the gauge anomaly associated with the Weyl fermions in the U(1) chiral lattice gauge theories,

$$q(x) = \mathrm{tr}\,\{Q\gamma_5(1-D)(x,x)\} \qquad (x \in \mathbb{Z}^4), \tag{2.44}$$

which is topological by virtue of the Ginsparg-Wilson relation [34, 49 – 53]:

Lemma 1. *The* U(1) *gauge anomaly* $q(x)$ *has the following form:*

$$q(x) = \gamma \left(\sum_\alpha e_\alpha^3 \right) \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x+\hat{\mu}+\hat{\nu}) + \partial^*_\mu \bar{k}_\mu(x), \tag{2.45}$$

where γ *is a constant and* $\bar{k}_\mu(x)$ *is a local, gauge-invariant current, which can be constructed so that it transforms as the axial vector current under the lattice symmetries. For the anomaly-free multiple, the cohomologically non-trivial part of the gauge anomaly cancels exactly at a finite lattice spacing and the total gauge anomaly is cohomologically trivial:*

$$q(x) = \partial^*_\mu \bar{k}_\mu(x). \tag{2.46}$$

[7] In fact, the second condition in eq. (2.43) follows from the first condition and the gauge invariance of $\Delta j_\mu(x)$ [36].

This result was shown in [35, 54, 55]. γ is a constant which takes the value $\gamma = \frac{1}{32\pi^2}$ for the overlap Dirac operator [49].

The second fact is about the representation of admissible link fields in terms of vector potentials with the desired locality property:

Lemma 2. *Suppose $U(x,\mu)$ is an admissible gauge field on the infinite lattice. Then there exists a vector potential $A_\mu(x)$ such that*

$$U(x,\mu) = \mathrm{e}^{iA_\mu(x)}, \quad |A_\mu(x)| \leq \pi(1+4\|x\|), \tag{2.47}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{2.48}$$

Moreover, any other field with these properties is equal to $A_\mu(x) + \partial_\mu\omega(x)$, where the gauge function $\omega(x)$ takes values that are integer multiples of 2π.

An important property of this mapping is that the locality properties of the gauge invariant fields are the same independently of whether they are considered to be functions of the link variables or the vector potential. To see this, let us first consider a local field which is composed from the link variables $U(x,\mu)$. Since the mapping $A_\mu(x) \to U(x,\mu) = \mathrm{e}^{iA_\mu(x)}$ is manifestly local, this function is local with respect to the vector potential. In the other direction, let us assume a gauge invariant local field $\phi(y)$ depending on the vector potential $A_\mu(x)$. Then we remind that it is free to change the gauge in constructing $A_\mu(x)$. In particular, we may impose a complete axial gauge taking the point y as the origin. Around y the vector potential $A_\mu(x)$ is locally constructed from the given link field $U(x,\mu)$. Thus $\phi(y)$ maps to a local function of the link variables $U(x,\mu)$ residing there.

Then, for a given admissible link field $U(x,\mu) = \mathrm{e}^{iA_\mu(x)}$ and any variational parameter $\eta_\mu(x)$ of compact support, one may define a linear functional $\mathfrak{L}^\star_\eta = \sum_{x\in\mathbb{Z}^4} \eta_\mu(x) j^\star_\mu(x)$ by the formula,

$$\begin{aligned}\mathfrak{L}^\star_\eta = i\int_0^1 ds\, \mathrm{Tr}\left\{\hat{P}_-\left[\partial_s\hat{P}_-, \delta_\eta\hat{P}_-\right]\right\} \\ + \int_0^1 ds \sum_{x\in\mathbb{Z}^4} \left\{\eta_\mu(x)\bar{k}_\mu(x) + A_\mu(x)\delta_\eta\bar{k}_\mu(x)\right\},\end{aligned} \tag{2.49}$$

where the differentiation and the integration with respect to the parameter s should be performed along the one-parameter family of the admissible link fields defined by $U_s(x,\mu) = \mathrm{e}^{isA_\mu(x)}$. This linear functional $\mathfrak{L}^\star_\eta$ satisfies all the properties required to the measure term in infinite volume. In particular, the current $j^\star_\mu(x)$ is a local functional of the link variables. To see this property, we first note that it is local with respect to the vector potential $A_\mu(x)$ because of the locality properties of the kernel of the projection operator $\hat{P}_-(x,y)$ and the current $\bar{k}_\mu(x)$. We next note that $j^\star_\mu(x)$ is invariant under the gauge transformations $A_\mu(x) \to A_\mu(x) + \partial_\mu\omega(x)$ for arbitrary gauge functions $\omega(x)$ that are polynomially bounded at infinity. Namely, taking the gauge covariance of $\hat{P}_-(x,y)$ and the gauge invariance of

$\hat{k}_\mu(x)$ into account, the change of $\mathfrak{L}^\star_\eta$ is evaluated as

$$\begin{aligned}
&\int_0^1 ds\,\mathrm{Tr}\big\{\hat{P}_-\big[[\omega Q,\hat{P}_-],\delta_\eta\hat{P}_-\big]\big\}+\int_0^1 ds\sum_{x\in\mathbb{Z}^4}\partial_\mu\omega(x)\,\delta_\eta\bar{k}_\mu(x)\\
&\qquad=-\int_0^1 ds\,\mathrm{Tr}\big\{\omega Q\,\delta_\eta\hat{P}_-\big\}+\int_0^1 ds\sum_{x\in\mathbb{Z}^4}\partial_\mu\omega(x)\,\delta_\eta\bar{k}_\mu(x)\\
&\qquad=\int_0^1 ds\sum_{x\in\mathbb{Z}^4}\omega(x)\,\delta_\eta\big\{-\mathrm{tr}\{Q\gamma_5 D\}(x,x)-\partial^*\bar{k}_\mu(x)\big\}=0, \qquad (2.50)
\end{aligned}$$

where the identity $\hat{P}_-\delta_\eta\hat{P}_-\hat{P}_-=0$ has been used. Then, we can regard $j^\star_\mu(x)$ as a local functional with respect to the link variables.

2.5.2 Second step in finite volume: smoothness

In the second step constructing the finite-volume correction $\mathfrak{S}_\eta$ which must be smooth with respect to the link variables, one needs to know the topological structure of the space of the admissible U(1) gauge fields in finite volume. It turns out that the space $\mathfrak{U}[m]$ is isomorphic to a multi-dimensional torus times a contractible space. Namely,

$$\mathfrak{U}[m]\cong \mathrm{U}(1)^4\times\mathfrak{G}_0\times\mathfrak{A}[m], \qquad (2.51)$$

where $\mathfrak{G}_0$ is the subset of the gauge transformations $\Lambda(x)\in\mathrm{U}(1)$ satisfying $\Lambda(x)=1$ at $x=0$ mod L, $\mathfrak{A}[m]$ is the space of the transverse vector potential $A^T_\mu(x)$ satisfying

$$\partial^*_\mu A^T_\mu(x)=0, \qquad \sum_{x\in\Gamma}A^T_\mu(x)=0, \qquad (2.52)$$

$$\big|\partial_\mu A^T_\nu(x)-\partial_\nu A^T_\mu(x)+2\pi m_{\mu\nu}/L^2\big|<\epsilon, \qquad (2.53)$$

and $\mathrm{U}(1)^4$ comes from the degrees of freedom of the Wilson lines. In fact, the following lemma provides a unique representation of $U(x,\mu)$ and establishes the isomorphism eq. (2.51) [36]:

Lemma 3. *The gauge fields $U(x,\mu)$ in the sector $\mathfrak{U}[m]$ are of the form*

$$U(x,\mu)=V_{[m]}(x,\mu)\;\mathrm{e}^{iA^T_\mu(x)}\;U_{[w]}(x,\mu)\,\Lambda(x)\,\Lambda(x+\hat{\mu})^{-1}, \qquad (2.54)$$

where $A^T_\mu(x)$ is the transverse vector potential in $\mathfrak{A}[m]$ satisfying

$$\partial_\mu A^T_\nu(x)-\partial_\nu A^T_\mu(x)+2\pi m_{\mu\nu}/L^2=F_{\mu\nu}(x), \qquad (2.55)$$

$U_{[w]}(x,\mu)$ represents the degrees of freedom of the Wilson lines,

$$U_{[w]}(x,\mu)=\begin{cases} w_\mu & \text{if } x_\mu=L-1,\\ 1 & \text{otherwise,}\end{cases} \qquad (2.56)$$

with the phase factor $w_\mu\in\mathrm{U}(1)$ and $\Lambda(x)$ is the gauge function in $\mathfrak{G}_0$ statisfying $\Lambda(0)=1$.

Once the topology of the space $\mathfrak{U}[m]$ is identified as a multi-dimensional torus times a constractible space, the construction of the smooth finite-volume correction $\mathfrak{S}_\eta$ is achieved based on the bound eq. (2.36) and the following mathematical fact:

Lemma 4. *Suppose T^n is the n-dimensional torus parameterized through $u = (\mathrm{e}^{it_1}, \mathrm{e}^{it_2}, \ldots, \mathrm{e}^{it_n})$ and $\mathfrak{C}_{kl}(t)$ is a smooth periodic tensor field on T^n satisfying*

$$\mathfrak{C}_{kl} = -\mathfrak{C}_{lk}, \qquad \partial_k \mathfrak{C}_{lj} + \partial_l \mathfrak{C}_{jk} + \partial_j \mathfrak{C}_{kl} = 0. \tag{2.57}$$

If the associated magnetic fluxes,

$$\mathfrak{I}_{kl} = \int_0^{2\pi} dt_k dt_l \, \mathfrak{C}_{kl}, \tag{2.58}$$

vanish, there exists smooth periodic vector field $\mathfrak{B}_k(t)$ such that $\mathfrak{C}_{kl} = \partial_k \mathfrak{B}_l - \partial_l \mathfrak{B}_k$ and

$$|\mathfrak{B}_k(t)| \leq \pi(n-1) sup_{r,k,l} |\mathfrak{C}_{kl}(r)| . \tag{2.59}$$

In fact, $\mathfrak{R}_{\eta\zeta}$ on the multi-dimensional torus $T^n \cong \mathrm{U}(1)^4 \times \mathfrak{G}_0$ turns out to satisfy all the premises of the above lemma and a solution of the integrability condition follows immediately from the lemma, which corresponds to the finite-volume correction term $\mathfrak{S}_\eta$ for $U(x,\mu) = V_{[m]}(x,\mu)\ U_{[w]}(x,\mu)\, \Lambda(x)\, \Lambda(x+\hat{\mu})^{-1}$ with the longitudinal variation $\eta^L_\mu(x)$:[8]

$$\eta^L_\mu(x) = L^{-4} \sum_{y \in \Gamma} \eta_\nu(y) + \sum_{y \in \Gamma} \partial_\mu G_L(x-y) \partial^*_\nu \eta_\nu(y). \tag{2.61}$$

It is then extended to the transverse degrees of freedom by the integration along the one-parameter family

$$U_t(x,\mu) = V_{[m]}(x,\mu)\ \mathrm{e}^{itA^T_\mu(x)}\ U_{[w]}(x,\mu)\, \Lambda(x)\, \Lambda(x+\hat{\mu})^{-1} \qquad (t \in [0,1]) \tag{2.62}$$

as follows:

$$\mathfrak{S}_\eta = \mathfrak{S}_{\eta^L}|_{t=0} + \int_0^1 dt\ \mathfrak{R}_{\zeta\eta}|_{\zeta_\mu = A^T_\mu}. \tag{2.63}$$

This completes the construction of the finite-volume correction term $\mathfrak{S}_\eta$.

3. A simple construction of the mesure term on the finite volume lattice

In the original construction by Lüscher [36], although the proof is constructive, the explicit formula of the measure term turns out to be complicated. In particular, it is based on the separate treatment of the part definable in infinite volume and the part of the finite volume corrections. Therefore it does not provide a formulation which is immediately usable for practical numerical applications.

[8] $G_L(z)$ is the Green function defined by

$$\partial^*_\mu \partial_\mu G_L(z) = \delta_{z,0} - L^{-4}, \quad \sum_{z \in \Gamma} G_L(z) = 0. \tag{2.60}$$

In this section, we describe our construction of the measure term on the finite volume lattice. We fisrt discuss the parametrization of the link fields in finite volume and their variations. We next state two useful results which hold true in finite volume: *the gauge anomaly cancellation in finite volume* and *the property of the curvature term for the Wilson lines*. Using these results, we write down a closed formula of the measure term directly within the finite volume theory. In our construction, the Wilson line degrees of freedom of the link field (torons) are treated separately to take care of the global integrability. The part of local counter term is then explicitly constructed in finite volume with the local current associated with the cohomologically trivial part of the gauge anomaly. Only in the final step to establish the locality property of the measure term current, we follow the procedure to separate the part definable in infinite volume from the part of the finite volume corrections as in the original construction [36].

3.1 Parametrization of the link fields and their variations in finite volume

In our construction of the measure term in finite volume, we adopt the parametrization of the link fields given by eq. (2.54). When a link field $U(x,\mu)$ is parameterized by eq. (2.54), the parametrization is unique and the each factors, $A^T_\mu(x)$, $U_{[w]}(x,\mu)$ and $\Lambda(x)$, may be regarded as the smooth functionals of the original link field $U(x,\mu)$.

Accordingly, the variation of the link field,

$$\delta_\eta U(x,\mu) = i\,\eta_\mu(x)\,U(x,\mu), \tag{3.1}$$

may be decomposed as follows:

$$\eta_\mu(x) = \eta^T_\mu(x) + \eta_{\mu[w]}(x) + \eta^\Lambda_\mu(x). \tag{3.2}$$

$\eta^T_\mu(x)$ is the transverse part of $\eta_\mu(x)$ defined by

$$\partial^*_\mu \eta^T_\mu(x) = 0, \qquad \sum_{x\in\Gamma} \eta^T_\mu(x) = 0, \tag{3.3}$$

which may be given explicitly as

$$\eta^T_\mu(x) = \sum_{y\in\Gamma} G_L(x-y)\partial^*_\lambda(\partial_\lambda\eta_\mu(x) - \partial_\mu\eta_\lambda(x)). \tag{3.4}$$

$\eta_{\mu[w]}(x)$ is the variation along the Wilson lines defined by

$$\eta_{\mu[w]}(x) = \sum_\nu \eta_{(\nu)}\,\delta_{\mu\nu}\,\delta_{x_\nu,L-1}, \qquad \eta_{(\nu)} = L^{-3}\sum_{y\in\Gamma}\eta_\nu(y). \tag{3.5}$$

$\eta^\Lambda_\mu(x)$ is the variation of the gauge degrees of freedom in the form,

$$\eta^\Lambda_\mu(x) = -\partial_\mu\omega_\eta(x), \qquad \omega_\eta(0) = 0. \tag{3.6}$$

This decomposition is also unique by the following reason: for an arbitrary periodic vector field $\eta_\mu(x)$, the vector field defined by $a_\mu(x) = \eta_\mu(x) - \eta^T_\mu(x) - \eta_{\mu[w]}(x)$ has the vanishing

field tensor $\partial_\mu a_\nu(x) - \partial_\nu a_\mu(x) = 0$ and the vanishing wilson lines $\sum_{s=0}^{L-1} a_\mu(x + s\hat{\mu}) = 0$. Then, the sum $\omega_\eta(x)$ of the vector field $a_\mu(x)$ along any lattice path from x to the origin $x = 0$ is independent of the chosen path, periodic in x and $\omega_\eta(0) = 0$. It gives the gauge function which reproduces $a_\mu(x)$ in the pure gauge form, $a_\mu(x) = -\partial_\mu \omega_\eta(x)$. This proves the uniqueness of the decomposition. The action of the differential operator δ_η to each factors, $A^T_\mu(x)$, $U_{[w]}(x,\mu)$ and $\Lambda(x)$, is then given as follows:

$$\delta_\eta A^T_\mu(x) = \eta^T_\mu(x), \tag{3.7}$$

$$\delta_\eta U_{[w]}(x,\mu) = i\,\eta_{\mu[w]}(x)\,U_{[w]}(x,\mu), \tag{3.8}$$

$$\delta_\eta \Lambda(x) = i\,\omega_\eta(x)\,\Lambda(x). \tag{3.9}$$

3.2 Useful results in finite volume

3.2.1 Gauge anomaly cancellation

In finite volume, the U(1) gauge anomaly is given by the formula,

$$q_L(x) = \mathrm{tr}\,\{Q\gamma_5(1 - D_L)(x,x)\} \qquad (x \in \Gamma), \tag{3.10}$$

which is topological [8, 13, 14] in the sense that

$$\sum_{x\in\Gamma} q_L(x) = \text{integer}. \tag{3.11}$$

For this gauge anomaly in finite volume, it is possible to establish the similar result as the lemma 1:

Lemma 5. *For the anomaly-free multiplet satisfying the condition eq. (2.1), the* U(1) *gauge anomaly $q_L(x)$ has the following form in sufficiently large volume L^4:*

$$q_L(x) = \partial^*_\mu k_\mu(x) \qquad (x \in \Gamma), \tag{3.12}$$

where $k_\mu(x)$ is a local, gauge-invariant current, which can be constructed so that it transforms as the axial vector current under the lattice symmetries.

This result was first obtained by combining the result in the infinite lattice, eq. (2.45) [35, 54, 55], and the result of the analysis of the finite volume correction $r(x)$ [56]. Namely,

$$k_\mu(x) = \bar{k}_\mu(x) + \Delta k_\mu(x), \tag{3.13}$$

where $\Delta k_\mu(x)$ satisfies

$$|\Delta k_\mu(x)| \le \kappa_3 L^{\nu_3} \mathrm{e}^{-L/\varrho} \tag{3.14}$$

for some constants $\kappa_3 > 0$, $\nu_3 \ge 0$ and

$$r(x) = \partial^*_\mu \Delta k_\mu(x) \qquad (x \in \Gamma) \tag{3.15}$$

in sufficiently large volume L^4. However, as shown in [57], it is possible to derive the same result directly from the gauge anomaly in finite volume $q_L(x)$ without the separate treatment of $q(x)$ and $r(x)$. This work also provides a procedure to work out the local current $k_\mu(x)$ explicitly, which can be implemented numerically [58].

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3.2.2 A solution of the integrability condition for the Wilson lines

The curvature terms associated with the Wilson lines have special properties which turn out to be useful in the construction of a solution of the integrability condition eq. (2.25). Let us parametrize the Wilson lines $U_{[w]}(x,\mu)$ defined by eq. (2.56) as

$$w_\mu = \exp(it_\mu), \quad t_\mu \in [0, 2\pi) \quad (\mu = 1,2,3,4), \tag{3.16}$$

and the variational parameters in the directions of the Wilson lines as

$$\lambda_{\mu(\nu)}(x) = \frac{1}{i}\,\partial_{t_\nu} U_{[w]}(x,\mu)\cdot U_{[w]}(x,\mu)^{-1} = \delta_{\mu\nu}\delta_{x_\nu, L-1}. \tag{3.17}$$

Then the curvature term for the Wilson lines reads

$$\begin{aligned} i\mathrm{Tr}_L\left\{\hat{P}_-[\delta_{\lambda_{(\mu)}}\hat{P}_-, \delta_{\lambda_{(\nu)}}\hat{P}_-]\right\}_{U=U_{[w]}V_{[m]}} &= i\mathrm{Tr}_L\left\{\hat{P}_-[\partial_{t_\mu}\hat{P}_-, \partial_{t_\nu}\hat{P}_-]\right\}_{U=U_{[w]}V_{[m]}} \\ &\equiv \mathfrak{C}_{\mu\nu}(t), \qquad t = (t_1,t_2,t_3,t_4). \end{aligned} \tag{3.18}$$

Then, the following lemma holds true:

Lemma 6. *In anomaly-free theories, the curvature term for the Wilson lines $\mathfrak{C}_{\mu\nu}(t)$, which possesses the properties*

$$\mathfrak{C}_{\mu\nu}(t) = -\mathfrak{C}_{\nu\mu}(t), \qquad \partial_\mu \mathfrak{C}_{\nu\rho}(t) + \partial_\nu \mathfrak{C}_{\rho\mu}(t) + \partial_\rho \mathfrak{C}_{\mu\nu}(t) = 0, \tag{3.19}$$

satisfies the bound

$$|\mathfrak{C}_{\mu\nu}(t)| \le \kappa_4 L^{\nu_4} \mathrm{e}^{-L/\varrho} \tag{3.20}$$

for certain positive constants κ_4 and ν_4. For a sufficiently large volume L^4, it then follows that

$$\int_0^{2\pi} dt_\mu \int_0^{2\pi} dt_\nu\ \mathfrak{C}_{\mu\nu}(t) = 0, \tag{3.21}$$

and there exists smooth periodic vector field $\mathfrak{W}_\mu(t)$ such that

$$\mathfrak{C}_{\mu\nu}(t) = \partial_\mu \mathfrak{W}_\nu(t) - \partial_\nu \mathfrak{W}_\mu(t), \qquad |\mathfrak{W}_\mu(t)| \le 3\pi\, sup_{t,\mu,\nu}\left|\mathfrak{C}_{\mu\nu}(t)\right|. \tag{3.22}$$

The proof of this lemma is based on the fact that in infinite-volume the periodic link field which represents the degrees of freedom of the Wilson lines can be written in the pure-gauge form,

$$U_{[w]}(x,\mu) = \Lambda_{[w]}(x)\Lambda_{[w]}(x+\hat{\mu})^{-1}, \quad \Lambda_{[w]}(x) = \prod_\mu (w_\mu)^{n_\mu} \quad \text{for } x - nL \in \Gamma, \tag{3.23}$$

and therefore the gauge-invariant function of the link field in infinite volume is actually independent of the degrees of freedom of the Wilson lines. In fact, from eqs. (2.32) and (2.36), $\mathfrak{C}_{\mu\nu}$ may be written as

$$\mathfrak{C}_{\mu\nu} = i\mathrm{Tr}\left\{Q_\Gamma \hat{P}_-[\delta_{\lambda_{(\mu)}}\hat{P}_-, \delta_{\lambda_{(\nu)}}\hat{P}_-]\right\} + \mathfrak{R}_{\lambda_{(\mu)}\lambda_{(\nu)}}, \tag{3.24}$$

where $\mathfrak{R}_{\lambda_{(\mu)}\lambda_{(\nu)}}$ satisfies the bound

$$\left|\mathfrak{R}_{\lambda_{(\mu)}\lambda_{(\nu)}}\right| \leq \kappa_4 L^{\nu_4} \mathrm{e}^{-L/\varrho} \tag{3.25}$$

for some constants $\kappa_4 > 0$, $\nu_4 \geq 0$ and $C_4 > 0$. We then recall the fact that there exists the measure term $\mathfrak{K}_\eta = \sum_{x\in\Gamma} \eta_\mu(x) j^\star_\mu(x)$ given by eq. (2.38), which satisfies the integrability condition eq. (2.39). The current $j^\star_\mu(x)$ is defined for all admissible gauge fields in infinite volume and it is local and gauge-invariant. Therefore, as discussed above, the current $j^\star_\mu(x)$ is actually independent of the Wilson lines and the curvature of $\mathfrak{K}_\eta$ evaluated in the directions of the Wilson lines vanishes identically. Namely,

$$i\mathrm{Tr}\left\{Q_\Gamma \hat{P}_-[\delta_{\lambda_{(\mu)}}\hat{P}_-, \delta_{\lambda_{(\nu)}}\hat{P}_-]\right\} = \delta_{\lambda_{(\mu)}}\mathfrak{K}_{\lambda_{(\nu)}} - \delta_{\lambda_{(\nu)}}\mathfrak{K}_{\lambda_{(\mu)}} = 0. \tag{3.26}$$

Then one can see that the curvature for the Wilson lines, $\mathfrak{C}_{\nu\lambda}$, itself satisfies the bound eq. (3.20) and because of this bound, the two-dimensional integration of the curvature, which should be a multiple of 2π, must vanish identically for a sufficiently large L. The existence of the smooth periodic vector field $\mathfrak{W}_\mu(t)$ then follows from the lemma 4 (the lemma 9.2 in [36]).

The properties of the curvature term $\mathfrak{C}_{\mu\nu}$ for the Wilson lines given by eq. (3.19) and (3.20) are useful because it implies that $\mathfrak{C}_{\mu\nu}$ itself satisfies the premise of the lemma 4 (the lemma 9.2 in [36]) and by using the lemma, one can construct a solution of the integrability condition,

$$\left\{\delta_{\lambda_{(\mu)}}\mathfrak{W}_\nu - \delta_{\lambda_{(\nu)}}\mathfrak{W}_\mu\right\}\Big|_{U=U_{[w]}V_{[m]}} = \mathfrak{C}_{\mu\nu}, \tag{3.27}$$

from $\mathfrak{C}_{\mu\nu}$ directly. Explicitly, it may be given by the formulae,

$$\begin{aligned}
\mathfrak{W}_4 &= \frac{1}{2\pi}\int_0^{2\pi} dr_4 \int_0^{(t_1,t_2,t_3)} \{dr_1\mathfrak{C}_{14} + dr_2\mathfrak{C}_{24} + dr_3\mathfrak{C}_{34}\},\\
\mathfrak{W}_3 &= \int_0^{t_4} dr_4\mathfrak{C}_{43} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{43} + \left[\frac{1}{2\pi}\int_0^{2\pi} dr_3 \int_0^{(t_1,t_2)} \{dr_1\mathfrak{C}_{13} + dr_2\mathfrak{C}_{23}\}\right]_{t_4=0},\\
\mathfrak{W}_2 &= \int_0^{t_4} dr_4\mathfrak{C}_{42} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{42}\\
&\quad + \left[\int_0^{t_3} dr_3\mathfrak{C}_{32} - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\mathfrak{C}_{32}\right]_{t_4=0} + \left[\frac{1}{2\pi}\int_0^{2\pi} dr_2 \int_0^{(t_1)} \{dr_1\mathfrak{C}_{12}\}\right]_{t_4=t_3=0},\\
\mathfrak{W}_1 &= \int_0^{t_4} dr_4\mathfrak{C}_{41} - \frac{t_4}{2\pi}\int_0^{2\pi} dr_4\mathfrak{C}_{41}\\
&\quad + \left[\int_0^{t_3} dr_3\mathfrak{C}_{31} - \frac{t_3}{2\pi}\int_0^{2\pi} dr_3\mathfrak{C}_{31}\right]_{t_4=0} + \left[\int_0^{t_2} dr_2\mathfrak{C}_{21} - \frac{t_2}{2\pi}\int_0^{2\pi} dr_2\mathfrak{C}_{21}\right]_{t_4=t_3=0}.
\end{aligned} \tag{3.28}$$

It follows from the properties of $\mathfrak{C}_{\mu\nu}$ that this solution is periodic and smooth with respect to the Wilson lines $U_{[w]}$ and satisfies the bound

$$|\,\mathfrak{W}_\nu\,| \leq \kappa_5 L^{\nu_5} \mathrm{e}^{-L/\varrho}, \tag{3.29}$$

for certain positive constants κ_5 and ν_5. It also follows that this measure term is gauge invariant. Then one may introduce the linear functional of the variational parameters in the directions of the Wilson lines $\eta_{\mu[w]}(x)$ at the gauge field $U(x,\mu) = U_{[w]}(x,\mu)V_{[m]}(x,\mu)$ by

$$\mathfrak{W}_\eta|_{U=U_{[w]}V_{[m]},\eta=\eta_{[w]}} = \sum_\nu \eta_{(\nu)}\mathfrak{W}_\nu. \tag{3.30}$$

This provides the measure term at the gauge field $U(x,\mu) = U_{[w]}(x,\mu)V_{[m]}(x,\mu)$.

3.3 A closed formula of the measure term

We now construct the measure term for the generic admissible $\mathrm{U}(1)$ gauge fields in the given topological sector $\mathfrak{U}[m]$. For this purpose, we introduce a vector potential defined by

$$\tilde{A}'_\mu(x) = A^T_\mu(x) - \frac{1}{i}\partial_\mu\Big[\ln\Lambda(x)\Big]; \qquad \frac{1}{i}\ln\Lambda(x) \in (-\pi,\pi], \tag{3.31}$$

and choose a one-parameter family of the gauge fields as

$$U_s(x,\mu) = \mathrm{e}^{is\tilde{A}'_\mu(x)}\, U_{[w]}(x,\mu)\, V_{[m]}(x,\mu), \qquad 0 \le s \le 1. \tag{3.32}$$

Then we consider the linear functional of the variational parameter $\eta_\mu(x)$, which is given in terms of quantities defined on the finite-volume lattice:

$$\begin{aligned}\mathfrak{L}^\diamond_\eta = {} & i\int_0^1 ds\,\mathrm{Tr}_L\left\{\hat{P}_-[\partial_s\hat{P}_-,\delta_\eta\hat{P}_-]\right\} \\ & +\delta_\eta\int_0^1 ds\sum_{x\in\Gamma}\left\{\tilde{A}'_\mu(x)\,k_\mu(x)\right\} + \mathfrak{W}_\eta|_{U=U_{[w]}V_{[m]},\eta=\eta_{[w]}},\end{aligned} \tag{3.33}$$

where $k_\mu(x)$ is the gauge-invariant local current which satisfies $\partial^*_\mu k_\mu(x) = q_L(x)$ and transforms as an axial vector field under the lattice symmetries. $\mathfrak{W}_\eta|_{U=U_{[w]}V_{[m]},\eta=\eta_{[w]}}$ is the additional measure term at the gauge field $U = U_{[w]}V_{[m]}$ with the variational parameters in the directions of the Wilson lines $\eta_{\mu[w]}(x)$. The current $j^\diamond_\mu(x)$ defined by eq. (3.33),

$$\mathfrak{L}^\diamond_\eta = \sum_{x\in\Gamma}\eta_\mu(x)j^\diamond_\mu(x), \tag{3.34}$$

may be regarded as a functional of the link variable $U(x,\mu)$ through the dependences on $\tilde{A}^T_\mu(x)$, $\Lambda(x)$ $(\ln\Lambda(x))$, $U_{[w]}(x,\mu)$ and $V_{[m]}(x,\mu)$. The action of the differential operator δ_η to the vector potential $\tilde{A}'_\mu(x)$ is evaluated as

$$\begin{aligned}\delta_\eta\tilde{A}'_\mu(x) &= \delta_\eta A^T_\mu(x) - \partial_\mu\left[\frac{1}{i}\{\delta_\eta\Lambda(x)\}\,\Lambda(x)^{-1}\right] = \eta^T_\mu(x) - \partial_\mu\omega_\eta(x) \\ &= \eta_\mu(x) - \eta_{\mu[w]}(x),\end{aligned} \tag{3.35}$$

and the variation of $U_s(x,\mu)$ is given by

$$\delta_\eta U_s(x,\mu) = i\left[s(\eta_\mu(x) - \eta_{\mu[w]}(x)) + \eta_{\mu[w]}(x)\right]U_s(x,\mu). \tag{3.36}$$

The linear functional so obtained, however, does not respect the lattice symmetries. (The first- and second- terms in the r.h.s. of eq. (3.33) transform properly, but the third term does not respect the lattice symmetries.) In order to make it to transform as a pseudo scalar field under the lattice symmetries, we should average it over the lattice symmetries with the appropriate weights so as to project to the pseudo scalar component. Namely, we take the average as follows:[9]

$$\bar{\mathfrak{L}}^{\diamond}_{\eta} = \frac{1}{2^4 4!} \sum_{R \in O(4,\mathbb{Z})} \det R \; \mathfrak{L}^{\diamond}_{\eta}|_{U \to \{U\}^{R^{-1}}, \eta_\mu \to \{\eta_\mu\}^{R^{-1}}} . \tag{3.37}$$

Our main result is then stated as follows:

Lemma 7. *The current $j^{\diamond}_{\mu}(x)$ defined by eq. (3.33),*

$$\mathfrak{L}^{\diamond}_{\eta} = \sum_{x \in \Gamma} \eta_\mu(x) j^{\diamond}_{\mu}(x),$$

fulfills all the properties required for the reconstruction theorem except the transformation property under the lattice symmetries. It may be corrected by invoking the average eq. (3.37) over the lattice symmetries with the appropriate weights so as to project to the pseudo scalar component.

3.3.1 Proof of the lemma 7

Although it is quite similar to that of theorem 5.3 in [36], we give the proof of the lemma 7 here for completeness.

1. *Smoothness.* By construction, $j^{\diamond}_{\mu}(x)$ is defined for all admissible gauge fields. It depends smoothly on $\tilde{A}'_{\mu}(x)$ and $U_{[w]}(x,\mu)$ because $\hat{P}_{-}$ and k_μ are smooth functions of $U_s(x,\mu)$. Although $\tilde{A}'_{\mu}(x)$ is not continuous when $\Lambda(x) = -1$ at some points x because of the cut in $\ln \Lambda(x)$, its discontinuity is always in the pure-gauge form

$$\mathrm{disc.}\{\tilde{A}'_{\mu}(x)\} = -\partial_\mu \omega(x); \qquad \omega(0) = 0, \tag{3.38}$$

where the gauge function $\omega(x)$ takes values that are integer multiples of 2π. Then, any smooth functionals of $\tilde{A}'_{\mu}(x)$ are smooth with respect to the link field $U(x,\mu)$, if they are gauge-invariant under the gauge transformations $\tilde{A}'_{\mu}(x) \to \tilde{A}'_{\mu}(x) + \partial_\mu \omega(x)$ for arbitrary periodic gauge functions $\omega(x)$ satisfying $\omega(0) = 0$. The current $j^{\diamond}_{\mu}(x)$ is indeed gauge-invariant under such gauge transformations. Namely, taking the gauge covariance of $\hat{P}_{-}(x,y)$ and the gauge invariance of $k_\mu(x)$ into account, the change of

[9] In doing the average, one should note the fact that under the lattice symmetries the Wilson lines $U_{[w]}(x,\mu)$ are transformed to other Wilson lines $U_{[w']}(x,\mu)$ *modulo gauge transformations*, $\{U_{[w]}(x,\mu)\}^{R^{-1}} = U_{[w']}(x,\mu)\Lambda(x)\Lambda(x+\hat{\mu})^{-1}$. Accordingly, the variational parameter $\eta_{\mu[w]}(x)$ is transformed as $\{\eta_{\mu[w]}(x)\}^{R^{-1}} = \eta_{\mu[w']}(x) - \partial_\mu \omega(x)$ with a certain periodic gauge funciton $\omega(x)$.

$\mathfrak{L}^{\diamond}_{\eta}$ under the gauge transformations is evaluated as

$$\int_0^1 ds\,\mathrm{Tr}\big\{\hat{P}_-\big[[\omega Q,\hat{P}_-],\delta_\eta\hat{P}_-\big]\big\}+\int_0^1 ds\sum_{x\in\Gamma}\partial_\mu\omega(x)\,\delta_\eta k_\mu(x) \tag{3.39}$$
$$=-\int_0^1 ds\,\mathrm{Tr}\big\{\omega Q\,\delta_\eta\hat{P}_-\big\}+\int_0^1 ds\sum_{x\in\Gamma}\partial_\mu\omega(x)\,\delta_\eta k_\mu(x)$$
$$=\int_0^1 ds\sum_{x\in\Gamma}\omega(x)\,\delta_\eta\left\{-\mathrm{tr}\{Q\gamma_5 D_L\}(x,x)-\partial^*_\mu k_\mu(x)\right\}=0,$$

where the identity $\hat{P}_-\delta_\eta\hat{P}_-\hat{P}_-=0$ has been used.

2. *Gauge invariance and symmetry properties.* The gauge invariance of $j^{\diamond}_\mu(x)$ has been shown above. The transformation properties of $j^{\diamond}_\mu(x)$ under the lattice symmetries are also evident from the average eq. (3.37).

3. *Integrability condition.* From the definition of $\mathfrak{L}^{\diamond}_{\eta}$, eq. (3.33), one finds immediately that the second term does not contribute the curvature $\delta_\eta\mathfrak{L}^{\diamond}_{\zeta}-\delta_\zeta\mathfrak{L}^{\diamond}_{\eta}$ and the third term gives the curevature term at the Wilson lines, $U=U_{[w]}V_{[m]}$, with the variational parameters $\eta,\zeta=\eta_{[w]}$. Taking the identity $\mathrm{Tr}_L\left\{\delta_1\hat{P}_-\delta_2\hat{P}_-\delta_3\hat{P}_-\right\}=0$ into account, the curvature is evaluated as

$$\begin{aligned}\delta_\eta\mathfrak{L}^{\diamond}_{\zeta}-\delta_\zeta\mathfrak{L}^{\diamond}_{\eta}&=i\int_0^1 ds\,\mathrm{Tr}\left\{\hat{P}_-[\delta_\eta\partial_s\hat{P}_-,\delta_\zeta\hat{P}_-]-\hat{P}_-[\delta_\zeta\partial_s\hat{P}_-,\delta_\eta\hat{P}_-]\right\}\\&\qquad+i\mathrm{Tr}\left\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\right\}\Big|_{U=U_{[w]}V_{[m]};\eta,\zeta=\eta_{[w]}}\\&=i\int_0^1 ds\,\partial_s\mathrm{Tr}\left\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\right\}\\&\qquad+i\mathrm{Tr}\left\{\hat{P}_-[\delta_\eta\hat{P}_-,\delta_\zeta\hat{P}_-]\right\}\Big|_{U=U_{[w]}V_{[m]};\eta,\zeta=\eta_{[w]}}.\end{aligned} \tag{3.40}$$

After the integration in the first term, the contribution from the lower end of the integration range exactly cancels with the second term because the variational parameters in this contribution is restricted to $\eta_{\mu[w]}(x)$:

$$\delta_\eta U_s(x,\mu)\,U_s(x,\mu)^{-1}|_{s=0}=[s(\eta_\mu(x)-\eta_{\mu[w]}(x))+\eta_{\mu[w]}(x)]_{s=0}=\eta_{\mu[w]}(x). \tag{3.41}$$

4. *Anomalous conservation law.* If one sets $\eta_\mu(x)=-\partial_\mu\omega(x)$ (where $\omega(x)$ is any lattice function on Γ with $\omega(0)=0$), the left-hand side of eq. (3.33) becomes

$$\sum_{x\in\Gamma}\omega(x)\,\partial^*_\mu j^{\diamond}_\mu(x). \tag{3.42}$$

On the other hand, using the identities

$$\delta_\eta\hat{P}_-=is\left[\omega Q,\hat{P}_-\right],\qquad \delta_\eta k_\mu(x)=0, \tag{3.43}$$

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the right-hand side is evaluated as

$$
\begin{aligned}
&-\int_0^1 ds\, s\,\mathrm{Tr}\{\omega Q\,\partial_s\hat{P}_-\} - \int_0^1 ds \sum_{x\in\Gamma}\partial_\mu\omega(x)\,k_\mu(x) \\
&= -\sum_{x\in\Gamma}\omega(x)\,\mathrm{tr}\{Q\gamma_5 D_L\}(x,x) + \int_0^1 ds \sum_{x\in\Gamma}\omega(x)\,\left\{\mathrm{tr}\{Q\gamma_5 D_L\}(x,x) + \partial_\mu^* k_\mu(x)\right\} \\
&= \sum_{x\in\Gamma}\omega(x)\,\mathrm{tr}\{Q\gamma_5(1-D_L)\}(x,x). \qquad (3.44)
\end{aligned}
$$

3.4 Locality property of the measure term

The locality property of the current $j_\mu^\diamond(x)$ may be examined by following the procedure to decompose the measure term eq. (3.33) into the part definable in infinite volume and the part of the finite volume corrections:

$$
\mathfrak{L}_\eta^\diamond = \mathfrak{K}_\eta^\diamond + \mathfrak{S}_\eta^\diamond, \qquad (3.45)
$$

where

$$
\begin{aligned}
\mathfrak{K}_\eta^\diamond &= i\int_0^1 ds\,\mathrm{Tr}\left\{Q_\Gamma\hat{P}_-[\partial_s\hat{P}_-,\delta_\eta\hat{P}_-]\right\} \\
&\qquad +\delta_\eta\int_0^1 ds\sum_{x\in\Gamma}\left\{\tilde{A}'_\mu(x)\,\bar{k}_\mu(x)\right\}, \qquad (3.46) \\
\mathfrak{S}_\eta^\diamond &= \int_0^1 ds\;\mathfrak{R}_{\zeta\eta}|_{\zeta_\mu=\tilde{A}'_\mu} \\
&\qquad +\delta_\eta\int_0^1 ds\sum_{x\in\Gamma}\left\{\tilde{A}'_\mu(x)\,\Delta k_\mu(x)\right\} + \mathfrak{L}_\eta|_{U=U_{[w]}V_{[m]},\eta=\eta_{[w]}}. \qquad (3.47)
\end{aligned}
$$

From eqs. (2.36), (3.14), (3.29) and $\|A_\mu^T(x)\| \le \kappa_6 L^4$ ($\kappa_6 > 0$) [36], one can infer

$$
\left|\mathfrak{S}_\eta^\diamond\right| \le \kappa_7 L^{\nu_7}\mathrm{e}^{-L/\varrho}\,\|\eta\|_\infty \qquad (3.48)
$$

for some constants $\kappa_7 > 0$, $\nu_7 \ge 0$.

As to $\mathfrak{K}_\eta^\diamond$ defined by eq. (3.46), if one introduces the truncated fields

$$
\eta_\mu^n(x) = \begin{cases} \eta_\mu(x) & \text{if } x - Ln \in \Gamma, \\ 0 & \text{otherwise,} \end{cases} \qquad (3.49)
$$

for any integer vector n, it may be rewritten into

$$
\begin{aligned}
\mathfrak{K}_\eta^\diamond &= i\int_0^1 ds\,\mathrm{Tr}\left\{P[\partial_s P,\delta_{\eta^0}P]\right\} \\
&\qquad +\int_0^1 ds\sum_{x\in\mathbb{Z}^4}\left\{(\eta_\mu^0(x)-\eta_{\mu[w]}^0(x))\,\bar{k}_\mu(x) + \tilde{A}'_\mu(x)\,\delta_{\eta^0}\,\bar{k}_\mu(x)\right\}. \qquad (3.50)
\end{aligned}
$$

One can see from this expression that $\mathfrak{K}_\eta^\diamond$ is defined in infinite volume for the variational parameter with a compact support $\eta_\mu^0(x)$. Then the following lemma holds ture:

Lemma 8. *$\mathfrak{K}_\eta^\diamond$ is in the form*

$$\mathfrak{K}_\eta^\diamond = \mathfrak{L}^\star_{\eta^0[m]}, \tag{3.51}$$

where $\mathfrak{L}^\star_{\eta[m]}$ is the linear functional defined in infinite volume for any variation parameter $\eta_\mu(x)$ with a compact support given by

$$\begin{aligned}\mathfrak{L}^\star_{\eta[m]} &= \int_0^1 ds \left[i\mathrm{Tr}\left\{P[\partial_s P, \delta_\eta P]\right\} + \sum_{x\in\mathbb{Z}^4} \left\{ \eta_\mu(x)\, \bar{k}_\mu(x) + \tilde{A}_\mu(x)\, \delta_\eta\, \bar{k}_\mu(x) \right\} \right]_{U_s = \mathrm{e}^{is\tilde{A}_\mu}\, V_{[m]}} \\ &\equiv \sum_{x\in\mathbb{Z}^4} \eta_\mu(x) j^\star_{\mu[m]}(x). \end{aligned} \tag{3.52}$$

$\tilde{A}_\mu(x)$ here is the vector potential which represents the dynamical degrees of freedom of the link field in the given topological sector $\mathfrak{U}[m]$, $\tilde{U}(x,\mu) = U(x,\mu)V_{[m]}(x,\mu)^{-1}$, with the following properties,

$$\begin{aligned} U(x,\mu) &= \mathrm{e}^{i\tilde{A}_\mu(x)} V_{[m]}(x,\mu), \quad |\tilde{A}_\mu(x)| \le \pi(1+4\|x\|), \\ F_{\mu\nu}(x) &= \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2}, \end{aligned} \tag{3.53}$$

and any other field with these properties is equal to $\tilde{A}_\mu(x) + \partial_\mu\omega(x)$, where the gauge function $\omega(x)$ takes values that are integer multiples of 2π.

The current $j^\star_{\mu[m]}(x)$ is quite similar in construction to $j^\star_\mu(x)$ defined by eq. (2.49) except the fact that $V_{[m]}(x,\mu)$ is chosen as the reference field at $s=0$. In particular, $j^\star_{\mu[m]}(x)$ is invariant under the gauge transformations $\tilde{A}_\mu(x) \to \tilde{A}_\mu(x) + \partial_\mu\omega(x)$ for arbitrary gauge functions $\omega(x)$ that are polynomially bounded at infinity. Then, the locality property of $j^\star_{\mu[m]}(x)$ can be established by the same argument as that given in [36], or in section 2.5.1.

3.4.1 Proof of the lemma 8

The proof of the lemma 8 may be given as follows. By noting eq. (3.23), we consider to change the one-parameter family of the link fields for the s-parameter integration by the shift of the vector potential,

$$\tilde{A}'_\mu(x) \longrightarrow \tilde{A}_\mu(x) = \tilde{A}'_\mu(x) - \partial_\mu \Omega_{[w]}(x)\ ; \qquad \Omega_{[w]}(x) = \frac{1}{i}\ln \Lambda_{[w]}(x), \tag{3.54}$$

so that the degrees of freedom of the Wilson lines are included in the vector potential. Since one may express $U_s(x,\mu)$ as

$$\begin{aligned} U_s(x,\mu) &= \mathrm{e}^{is\tilde{A}'_\mu(x)}\, \mathrm{e}^{i\Omega_{[w]}(x)} \mathrm{e}^{-i\Omega_{[w]}(x+\hat{\mu})}\, V_{[m]}(x,\mu) \\ &= \mathrm{e}^{is\tilde{A}_\mu(x)}\, \mathrm{e}^{i(1-s)\Omega_{[w]}(x)} \mathrm{e}^{-i(1-s)\Omega_{[w]}(x+\hat{\mu})}\, V_{[m]}(x,\mu). \end{aligned} \tag{3.55}$$

Accordingly, we have

$$\begin{aligned} \partial_s P\big|_{U_s = \mathrm{e}^{is\tilde{A}'_\mu}\, U_{[w]}\, V_{[m]}} &= \Lambda_{[w]}^{(1-s)} \left\{ \partial_s P - i[\Omega_{[w]} Q, P] \right\}_{U_s = \mathrm{e}^{is\tilde{A}_\mu}\, V_{[m]}} \Lambda_{[w]}^{-(1-s)}, \\ \delta_\eta P\big|_{U_s = \mathrm{e}^{is\tilde{A}'_\mu}\, U_{[w]}\, V_{[m]}} &= \Lambda_{[w]}^{(1-s)} \left\{ \delta_\eta P + i(1-s)[\delta_\eta \Omega_{[w]} Q, P] \right\}_{U_s = \mathrm{e}^{is\tilde{A}_\mu}\, V_{[m]}} \Lambda_{[w]}^{-(1-s)}. \end{aligned} \tag{3.56}$$

Then, the first term in the r.h.s. of eq. (3.50) reads

$$
\begin{aligned}
i\int_0^1 ds\,\mathrm{Tr}\left\{P[\partial_s P,\delta_{\eta^0}P]\right\} &= i\int_0^1 ds\,\mathrm{Tr}\left\{P[\partial_s P,\delta_{\eta^0}P]\right\}|_{U_s=\mathrm{e}^{is\tilde{A}_\mu}\,V_{[m]}} \\
&\quad+\int_0^1 ds\,\mathrm{Tr}\left\{Q_\Gamma P[[\Omega_{[w]}Q,P],\delta_\eta P]]\right\}|_{U_s=\mathrm{e}^{is\tilde{A}_\mu}\,V_{[m]}} \\
&\quad-\int_0^1 ds\,(1-s)\,\mathrm{Tr}\left\{Q_\Gamma P[\partial_s P,[\delta_\eta\Omega_{[w]}Q,P]]\right\}|_{U_s=\mathrm{e}^{is\tilde{A}_\mu}\,V_{[m]}} \\
&\quad+i\int_0^1 ds(1-s)\mathrm{Tr}\left\{Q_\Gamma P[[\Omega_{[w]}Q,P],[\delta_\eta\Omega_{[w]}Q,P]]\right\}|_{U_s=\mathrm{e}^{is\tilde{A}_\mu}V_{[m]}}.
\end{aligned}
\tag{3.57}
$$

In the r.h.s. of eq. (3.57), the second term may be evaluated as follows:

$$
\begin{aligned}
\int_0^1 ds\,\mathrm{Tr}\left\{Q_\Gamma P[[\Omega_{[w]}Q,P],\delta_\eta P]]\right\} &= \int_0^1 ds\,\mathrm{Tr}\left\{P[[\Omega_{[w]}Q,P],\delta_{\eta^0}P]]\right\} \\
&= -\int_0^1 ds\,\mathrm{Tr}\left\{\Omega_{[w]}Q\,\delta_{\eta^0}P\right\} \\
&= \int_0^1 ds\sum_{x\in\mathbb{Z}^4}\left\{(-\partial_\mu\Omega_{[w]}(x))\,\delta_{\eta^0}\,\bar{k}_\mu(x)\right\}.
\end{aligned}
\tag{3.58}
$$

In this evaluation, we should note that although $\Omega_{[m]}(x)=\sum_\mu n_\mu \ln w_\mu$ $(x-nL\in\Gamma)$ is not periodic, the operator $P[[\Omega_{[w]}Q,P],\delta_\eta P]]$ is translational invariant,

$$
P[[\Omega_{[w]}Q,P],\delta_\eta P]](x,y)=P[[\Omega_{[w]}Q,P],\delta_\eta P]](x+n_0L,y+n_0L) \tag{3.59}
$$

for any constant integer vector n_0, because the shift in $\Omega_{[w]}(x)$, $\Omega_{[w]}(x+n_0L)-\Omega_{[w]}(x)$, is independent of x and does not contribute to the operator. As to the third term in the r.h.s. of eq. (3.57), we introduce the truncation of the s-differential as

$$
(\partial_s)^n U_s(x,\mu)=\begin{cases} i\tilde{A}_\mu(x)\,U_s(x,\mu) & \text{if } x-Ln\in\Gamma,\\ 0 & \text{otherwise,}\end{cases} \tag{3.60}
$$

for any integer vector n. Then, it may be evaluated as follows:

$$
\begin{aligned}
&-\int_0^1 ds\,(1-s)\,\mathrm{Tr}\left\{Q_\Gamma P[\partial_s P,[\delta_\eta\Omega_{[w]}Q,P]]\right\} \qquad (3.61)\\
&\quad=-\int_0^1 ds\,(1-s)\,\mathrm{Tr}\left\{P[(\partial_s)^0 P,[\delta_\eta\Omega_{[w]}Q,P]]\right\} \\
&\quad=-\int_0^1 ds\,(1-s)\,\mathrm{Tr}\left\{\delta_\eta\Omega_{[w]}Q\,(\partial_s)^0 P\right\} \\
&\quad=\int_0^1 ds\,(1-s)\sum_{x\in\mathbb{Z}^4}\left\{\delta_\eta(-\partial_\mu\Omega_{[w]}(x))\,(\partial_s)^0\bar{k}_\mu(x)\right\} \\
&\quad=\int_0^1 ds\,(1-s)\sum_{x\in\Gamma}\eta_{\mu[w]}(x)\,\partial_s\bar{k}_\mu(x) \\
&\quad=\int_0^1 ds\sum_{x\in\mathbb{Z}^4}\eta^0_{\mu[w]}(x)\,\bar{k}_\mu(x)\,|_{U_s=\mathrm{e}^{is\tilde{A}_\mu}\,V_{[m]}}-\sum_{x\in\mathbb{Z}^4}\eta^0_{\mu[w]}(x)\,\bar{k}_\mu(x)\,|_{U=V_{[m]}}.
\end{aligned}
$$

In the last two steps above, we have used the relation

$$\delta_\eta(-\partial_\mu\Omega_{[w]}(x)) = (1/i)\,\delta_\eta U_{[w]}(x,\mu)\,U_{[w]}(x,\mu)^{-1} = \eta_{\mu[w]}(x), \tag{3.62}$$

and the fact that the local, gauge-invariant current $\bar{k}_\mu(x)$ at the link field $U(x,\mu) = V_{[m]}(x,\mu)$ with the constant field tensor is independent of x and may be set to zero:

$$\bar{k}_\mu(x)\big|_{U=V_{[m]}} \equiv 0. \tag{3.63}$$

The fourth term in the r.h.s. of eq. (3.57) turns out to vanish identically: by noting the hermiticity of $P(x,y)$, this term reads

$$\begin{aligned} & i\int_0^1 ds\,(1-s)\,\mathrm{Tr}\left\{Q_\Gamma P[[\Omega_{[w]}Q,P],[\delta_\eta\Omega_{[w]}Q,P]]\right\} \\ &\quad = i\int_0^1 ds\,(1-s)\,\frac{1}{2}\mathrm{Tr}\left\{Q_\Gamma\left(P\Omega_{[w]}QP\delta_\eta\Omega_{[w]}QP - P\delta_\eta\Omega_{[w]}QP\Omega_{[w]}QP\right.\right. \\ &\qquad\qquad \left.\left. -P\Omega_{[w]}Q\,\delta_\eta\Omega_{[w]}QP + P\delta_\eta\Omega_{[w]}Q\,\Omega_{[w]}QP\right)\right\} \\ &\quad = i\int_0^1 ds\,(1-s)\,\frac{1}{2}\sum_{\mu\nu}[(1/i)\ln w_\mu]\,\eta_{(\nu)}\;\left[I_{\mu\nu} - I_{\nu\mu} - J_{\mu\nu} + J_{\nu\mu}\right], \end{aligned} \tag{3.64}$$

where

$$I_{\mu\nu} = \sum_{x,y,z\in\Gamma}\sum_{n,n'\in\mathbb{Z}^4}\mathrm{tr}\left\{P(x,y+nL)n_\mu P(y+nL,z+n'L)n'_\nu P(z+n'L,x)\right\}, \tag{3.65}$$

$$J_{\mu\nu} = \sum_{x,y\in\Gamma}\sum_{n\in\mathbb{Z}^4}\mathrm{tr}\left\{P(x,y+nL)n_\mu n_\nu P(y+nL,x)\right\}, \tag{3.66}$$

but, $I_{\mu\nu}$ and $J_{\mu\nu}$ are both symmetric with respect to the indices μ,ν. Combining these results and using the gauge invariance of $\bar{k}_\mu(x)$,

$$\bar{k}_\mu(x)\big|_{U_s=\mathrm{e}^{is\tilde{A}'_\mu}\,V_{[m]}} = \bar{k}_\mu(x)\big|_{U_s=\mathrm{e}^{is\tilde{A}_\mu}\,V_{[m]}}, \tag{3.67}$$

in the second term of the r.h.s. of eq. (3.50), we finally obtain

$$\mathfrak{K}^\diamond_\eta = \int_0^1 ds\left[i\mathrm{Tr}\left\{P[\partial_s P,\delta_{\eta^0}P]\right\} + \sum_{x\in\mathbb{Z}^4}\left\{\eta^0_\mu(x)\,\bar{k}_\mu(x) + \tilde{A}_\mu(x)\,\delta_{\eta^0}\,\bar{k}_\mu(x)\right\}\right]_{U_s=\mathrm{e}^{is\tilde{A}_\mu}\,V_{[m]}}. \tag{3.68}$$

As the final step, we cast the vector potential $\tilde{A}_\mu(x)$, which represents

$$\tilde{U}(x,\mu) = \mathrm{e}^{iA^T_\mu(x)}\Lambda(x)\,\Lambda(x+\hat{\mu})^{-1}\,U_{[w]}(x,\mu) = U(x,\mu)V_{[m]}(x,\mu)^{-1}, \tag{3.69}$$

the dynamical degrees of freedom of the link field in the given topological sector $\mathfrak{U}[m]$, into the complete axial gauge with the properties eq. (3.53), by applying the same construction as in the lemma 2 [35] to $\tilde{U}(x,\mu)$. Since $\mathfrak{K}^\diamond_\eta$ is invariant under the gauge transformation $\tilde{A}_\mu(x) \to \tilde{A}_\mu(x)+\partial_\mu\omega(x)$ for arbitrary gauge functions $\omega(x)$ that are polynomially bounded at infinity, as easily verified, this change of the gauge for $\tilde{A}_\mu(x)$ does not alter $\mathfrak{K}^\diamond_\eta$ itself. This results in eq. (3.51) of the lemma 8.

4. Discussion

We have given a closed formula eq. (3.33) of the measure term on the finite volume lattice, which fulfills all the required properties for the reconstruction theorem in the gauge-invariant formulation of U(1) chiral gauge theories [36]. Although it is intended for the use in a practical implementation of U(1) chiral lattice gauge theories, it also provides, we believe, a simpler point of view on the theoretical structure of the formulation.

A comment is in order about the relation between the measure term $\mathfrak{L}^{\diamond}_{\eta}$ constructed in this paper and the measure term $\mathfrak{L}_{\eta}$ given in the original construction [36]. Since both terms satisfy the integrability condition and the anomalous conservation law, one may expect that they are related each other by the variation of a certain gauge-invariant local term as

$$\mathfrak{L}^{\diamond}_{\eta} = \mathfrak{L}_{\eta} + \sum_{x\in\Gamma} \delta_{\eta}\mathfrak{D}(x). \tag{4.1}$$

In fact, as to the terms definable in infinite volume, it is possible to work out the difference between the linear functionals $\mathfrak{K}^{\diamond}_{\eta^0} = \mathfrak{L}^{\star}_{\eta^0[m]}$ and $\mathfrak{K}_{\eta^0} = \mathfrak{L}^{\star}_{\eta^0}$ explicitly and the result is given in the following form:

$$\mathfrak{L}^{\star}_{\eta^0[m]} = \mathfrak{L}^{\star}_{\eta^0} + \sum_{x\in\mathbb{Z}^4} \delta_{\eta^0}\mathfrak{D}^{\star}_{[m]}(x), \tag{4.2}$$

where $\mathfrak{D}^{\star}_{[m]}(x)$ is the local field given by

$$\begin{aligned}\mathfrak{D}^{\star}_{[m]}(x) = \int_0^1 dt \int_0^1 ds \Big[& i\,\mathrm{tr}\,\{P[\partial_t P, \partial_s P]\}\,(x,x) \\ & +(s\tilde{A}_\mu+[A_\mu(x)-\tilde{A}_\mu(x)])\partial_s\bar{k}_\mu(x)-t\tilde{A}_\mu(x)\partial_t\bar{k}_\mu(x)\Big]_{U_{t,s}=\mathrm{e}^{it(s\tilde{A}+[A-\tilde{A}])}}.\end{aligned} \tag{4.3}$$

The relation eq. (4.1) implies that the resulted Weyl fermion measures, or the effective actions induced by the Weyl fermion path integral, differ by the gauge-invariant local term $\sum_{x\in\Gamma}\mathfrak{D}(x)$. We do not know, however, if there exists a closed expression of $\mathfrak{D}(x)$ in terms of only the quantities defined in finite volume like eq. (3.33) for $\mathfrak{L}^{\diamond}_{\eta}$.

In the formula of the measure term, eq. (3.33), we have adopted the transverse gauge for the vector potential $\tilde{A}'_\mu(x) = A^T_\mu(x) - \frac{1}{i}\partial_\mu\Big[\ln\Lambda(x)\Big]$ such that

$$\begin{aligned}\mathrm{e}^{i\tilde{A}'_\mu(x)} &= \mathrm{e}^{iA^T_\mu(x)}\Lambda(x)\Lambda(x+\hat{\mu})^{-1} \\ &= U(x,\mu)\,V_{[m]}(x,\mu)^{-1}\,U_{[w]}(x,\mu)^{-1} \equiv \tilde{U}'(x,\mu).\end{aligned} \tag{4.4}$$

Since the measure term (current) is gauge-invariant, one may choose different gauge conditions. For example, one may adopt the complete axial gauge, inspired by the following lemma (L is assumed to be an even number):

Lemma 9. *There exists a periodic vector potential $\tilde{A}'_\mu(x)$ such that*

$$\mathrm{e}^{i\tilde{A}'_\mu(x)} = \tilde{U}'(x,\mu), \tag{4.5}$$

$$\partial_\mu \tilde{A}'_\nu(x) - \partial_\nu \tilde{A}'_\mu(x) = F_{\mu\nu}(x) - \frac{2\pi m_{\mu\nu}}{L^2}, \tag{4.6}$$

$$\sum_{s=1}^{L} \tilde{A}'_\mu(x + s\hat{\mu}) = \frac{1}{i} \ln \left[\prod_{s=1}^{L} \tilde{U}'(x + s\hat{\mu}, \mu) \right], \tag{4.7}$$

and

$$\begin{cases} |\tilde{A}'_\mu(x)| \le \pi(1 + 4\|x - x_0\|) & \text{for } \ |(x - x_0)_\nu| \neq L/2 - 1 \ (\nu = 1, 2, 3, 4), \\ |\tilde{A}'_\mu(x)| \le \pi(1 + 6L^2 + 2L(1 + 3L^2)) & \text{otherwise.} \end{cases} \tag{4.8}$$

x_0 is a reference point which may be chosen arbitrarily. Moreover, if $\tilde{A}''_\mu(x)$ is any other field with these properties we have

$$\tilde{A}''_\mu(x) = \tilde{A}'_\mu(x) + \partial_\mu \omega(x), \tag{4.9}$$

where the gauge function $\omega(x)$ is periodic *and takes values that are integer multiples of 2π.*

The proof of this lemma is given in appendix A. Note that the variation of the vector potential in this gauge is also given by $\delta_\eta \tilde{A}'_\mu(x) = \eta_\mu(x) - \eta_{\mu[w]}(x)$.

Given the local current $j^\diamond_\mu(x)$, the basis vectors of the Weyl field can be constructed explicitly as follows [37]:

$$v_j(x) = \begin{cases} Q_1 v_{1[m]}\, W^{-1} & \text{if } j = 1, \\ Q_1 v_{j[m]} & \text{otherwise,} \end{cases} \tag{4.10}$$

where, along the one-parameter family of the link fields in $\mathfrak{U}[m]$,

$$U_t(x, \mu) = \mathrm{e}^{it\tilde{A}'_\mu(x)}\, U_{[w]}(x, \mu)\, V_{[m]}(x, \mu), \qquad 0 \le t \le 1, \tag{4.11}$$

W is defined by

$$W \equiv \exp\left\{ i \int_0^1 dt\, \mathfrak{L}^\diamond_\eta \right\}, \qquad \eta_\mu(x) = i\partial_t U_t(x, \mu)\, U_t(x, \mu)^{-1}, \tag{4.12}$$

Q_t is defined by the evolution operator of the projector $P_t = \hat{P}_-\Big|_{U=U_t}$ satisfying

$$\partial_t Q_t = [\partial_t P_t, P_t]\, Q_t, \quad Q_0 = 1, \tag{4.13}$$

and $v_{j[m]}$ are the basis vectors at the Wilson lines $U_{[w]}(x, \mu)V_{[m]}(x, \mu)$ $(t = 0)$.[10] Towards a numerical application of U(1) chiral lattice gauge theories, a next step is the practical implementation of this formula: a computation of W, the implementation of the operator Q_t and the construction of $v_{j[m]}$. This question has been addressed partly in our previous works [58, 47, 48]. We will disscuss this question in full detail elsewhere.

[10] With this definition, the explicit expression of W is given by

$$W = \exp\left\{ i \int_0^1 ds \sum_{x \in \Gamma} \tilde{A}'_\mu(x)\ k_\mu(x)|_{\tilde{A}' \to s\tilde{A}'} \right\}. \tag{4.14}$$

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A. Proof of the lemma 9

For simplicity, we set the reference point to the origin, $x_0 = 0$. The extention of the following proof to the case with a generic reference point x_0 is straightforward.

We introduce a vector potential

$$\tilde{a}_\mu(x) = \frac{1}{i} \ln \left[\tilde{U}'(x,\mu) \right], \qquad -\pi < \tilde{a}_\mu(x) \le \pi \tag{A.1}$$

where

$$\tilde{U}'(x,\mu) = \mathrm{e}^{iA_\mu^T(x)}\, \Lambda(x)\Lambda(x+\hat{\mu})^{-1} = U(x,\mu)\, V_{[m]}(x,\mu)^{-1}\, U_{[w]}(x,\mu)^{-1}, \tag{A.2}$$

and then note that

$$F_{\mu\nu}(x) = \partial_\mu \tilde{a}_\nu(x) - \partial_\nu \tilde{a}_\mu(x) + \frac{2\pi m_{\mu\nu}}{L^2} + 2\pi \tilde{n}_{\mu\nu}(x), \tag{A.3}$$

where $\tilde{n}_{\mu\nu}(x)$ is an anti-symmetric tensor field with integer values which satisfies

$$\partial_{[\rho} \tilde{n}_{\mu\nu]}(x) = 0, \tag{A.4}$$

$$\sum_{s,t=0}^{L-1} \tilde{n}_{\mu\nu}(x + s\hat{\mu} + t\hat{\nu}) = 0. \tag{A.5}$$

The Bianci identity of $\tilde{n}_{\mu\nu}(x)$ follows from the Bianci identity of $F_{\mu\nu}(x)$ which holds true for $\epsilon < \pi/3$.

We now construct a *periodic* integer vector field $\tilde{m}_\mu(x)$ such that $\partial_\mu \tilde{m}_\nu - \partial_\nu \tilde{m}_\mu = \tilde{n}_{\mu\nu}$. For this purpose, we try to impose a complete axial gauge where $\tilde{m}_1(x) = 0$, $\tilde{m}_2(x)|_{x_1=0} = 0$, $\tilde{m}_3(x)|_{x_1=x_2=0} = 0$, $\tilde{m}_4(x)|_{x_1=x_2=x_3=0} = 0$ and to obtain the non-zero components of the field by solving

$$\partial_\mu \tilde{m}_\nu(x) = \tilde{n}_{\mu\nu}(x) \quad at \quad x_1 = \cdots = x_{\mu-1} = 0 \tag{A.6}$$

for $\mu = 3, 2, 1$ (in this order) and $\nu > \mu$. However, the resulted vector potential is not periodic. Let us denote the restriction of the solution on to Γ by $m_\mu(x)$,

$$m_\mu(x) = -\sum_{\nu<\mu} {\sum_{t_\nu=0}^{x_\nu-1}}' \, \tilde{n}_{\mu\nu}(z^{(\nu)})\Big|_{x_1=\cdots=x_{\nu-1}=0} \tag{A.7}$$

where $x \in \Gamma$, $z^{(\nu)} = (x_1, \ldots, t_\nu, \ldots)$ and

$${\sum_{t_i=0}^{x_i-1}}' f(x) = \begin{cases} \sum_{t_i=0}^{x_i-1} f(x) & (x_i \ge 1) \\ 0 & (x_i = 0) \\ \sum_{t_i=x_i}^{-1} (-1) f(x) & (x_i \le -1) \end{cases}. \tag{A.8}$$

Although it satisfies the bound $|m_\mu(x)| \leq 2 \parallel x \parallel$, it only satisfies

$$\tilde{n}_{\mu\nu} = \partial_\mu m_\nu - \partial_\nu m_\mu + \Delta\tilde{n}_{\mu\nu}, \tag{A.9}$$

$$\Delta\tilde{n}_{\mu\nu}(x) = \delta_{x_\mu, L/2-1} \sum_{\tilde{t}_\mu=0}^{L-1} \tilde{n}_{\mu\nu}(z^{(\mu)})\Big|_{x_1=\cdots=x_{\mu-1}=0}, \tag{A.10}$$

where $\nu > \mu$ and $\tilde{t}_\mu = t_\mu \bmod L$. We note that $\Delta\tilde{n}_{\mu\nu}(x)$ has the support on the boundary of Γ. We then use the lattice counterpart of the lemma 9.2 in [36], to obtain the periodic integer vector potential $\Delta m_\mu(x)$ which solve $\partial_\mu \Delta m_\nu - \partial_\nu \Delta m_\mu = \Delta\tilde{n}_{\mu\nu}$,

$$\Delta m_\mu(x) = \delta_{x_\mu, L/2-1} \sum_{\nu>\mu} \sum_{t_\nu=0}^{x_\nu-1}{}' \sum_{\tilde{t}_\mu=0}^{L-1} \tilde{n}_{\mu\nu}(z^{(\mu,\nu)})\Big|_{x_{\nu+1}=\cdots=0}, \tag{A.11}$$

The desired periodic integer vector potential $\tilde{m}_\mu(x)$ is now obtained by $\tilde{m}_\mu(x) = m_\mu(x) + \Delta m_\mu(x)$, which satisfies the bound

$$\begin{cases} |\tilde{m}_\mu(x)| \leq 2 \parallel x \parallel \text{ for } \ x_\nu \neq L/2-1 \ (\nu = 1,2,3), \\ |\tilde{m}_\mu(x)| \leq 3L^2 \qquad \text{otherwise.} \end{cases} \tag{A.12}$$

Finally, we note that the differences of the Wilson lines between $\tilde{U}'(x,\mu)$ and $\tilde{a}_\mu(x) + 2\pi\tilde{m}_\mu(x)$ are integer multiples of 2π. Namely, one has

$$\frac{1}{i} \ln \left[\prod_{s=1}^{L} \tilde{U}'(x + s\hat{\mu}, \mu) \right] - \sum_{s=1}^{L} \{\tilde{a}_\mu(x + s\hat{\mu}) + 2\pi\tilde{m}_\mu(x + s\hat{\mu})\} = 2\pi c_\mu(x), \tag{A.13}$$

where $c_\mu(x) (\mu = 1,2,3,4)$ take integer values. One can also infer the bound

$$|c_\mu(x)| \ \leq \ L(1 + 3L^2). \tag{A.14}$$

Then, the vector potential with the desired properties is obtained by

$$\tilde{A}'_\mu(x) = \tilde{a}_\mu(x) + 2\pi\tilde{m}_\mu(x) + \delta_{x_\mu, L/2-1}\, 2\pi c_\mu(x). \tag{A.15}$$

If there exists a vector potential $\tilde{A}''_\mu(x)$ with the same properties as $\tilde{A}'_\mu(x)$, the vector potential defined by the difference $\tilde{A}''_\mu(x) - \tilde{A}'_\mu(x)$ has the vanishing field tensor and the vanishing Wilson lines. Such a vector potential should be in pure gauge form,

$$\tilde{A}''_\mu(x) - \tilde{A}'_\mu(x) = \partial_\mu \omega(x), \tag{A.16}$$

where the gauge function $\omega(x)$ is *periodic* and takes values that are integer multiples of 2π.

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